Back Paper
Classical Mechanics

B. Math., 2^{nd} Year, July - November 2024,
Date: December 27, 2024 (Friday)
Total points: 60.
Total time: 3 hours

Provide arguments for steps which are not obvious. Any result written down without explanations will receive zero credit.

1. A uniform disc of mass M and radius R rolls without slipping on a horizontal plane and is also attracted to a point P below the plane. This attractive force is proportional to the distance from the center of the disc to P and is oriented along the line joining the center of the disc to P. The vertical distance of P from the plane is d. Find the frequency of small oscillations of the disc about the point of equilibrium.

   10 points

2. (a) Consider a thin homogeneous plate that lies in the $x_1 - x_2$ plane. Show that in the $x_1 - x_2 - x_3$ axes, the inertia tensor takes the form

$$I = \begin{bmatrix} A & -C & 0 \\ -C & B & 0 \\ 0 & 0 & A+B \end{bmatrix}$$

   (b) The co-ordinate axes are rotated by an angle θ about the x_3 axis. Find the new moment of inertia tensor in terms A, B, C and θ. At which value of θ does the new $x_1 - x_2$-axes become the principal axes of inertia?

   10 + 10 = 20 points

3. A particle of mass m moves in one dimension under the influence of a force

$$F(x,t) = \frac{k}{x^2}\exp(-t/\tau)$$

   where k and τ are positive constants.

   (a) Find the Lagrangian and the Hamiltonian functions of the problem.

(b) Discuss energy conservation in the problem by comparing the Hamiltonian and total energy.

6 + 4 = 10 points

4. A sphere of mass m and radius ρ is constrained to roll down without slipping (under gravity) on the lower half of the inside surface of a hollow cylinder of radius $R (R > \rho)$. Find the frequency of small oscillations about the point of equilibrium.

   10 points

5. A particle of mass m rests on a smooth horizontal plane at $t = 0$. The inclination of the plane is changed at a constant rate α, i.e., $\theta(t) = \alpha t$, $\alpha > 0$. This causes the particle to move down under gravity.

   (a) Find the equation of the motion of the particle.

   (b) Breaking up the solution of the resulting ODE in its homogeneous and particular parts, and using appropriate initial conditions as given, find the motion of the particle at $t > 0$.

   3 + 7 = 10 points